§3.2 Mean Value Theorem (MVT)

Mean Value Theorem (MVT): If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ that satisfies $f'(c) = \frac{f(b)-f(a)}{b-a}$.

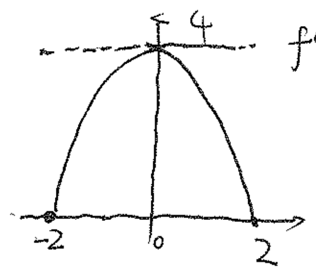
key points: ① The requirements for f: continuous on the CLOSED interval $[a,b]$ and differentiable on the OPEN interval (a,b).

② Geometric meaning of MVT.

③ Find (Solve for) c in MVT.

e.g. 1. (Baby version, Rolle's Theorem).

Consider the function $f(x) = 4-x^2$ on $[-2,2]$. Sketch the graph of f.
Apply MVT to f with $a=-2, b=2$. What does MVT tell you in the graph?



$f(-2) = f(2) = 4-4 = 0$.

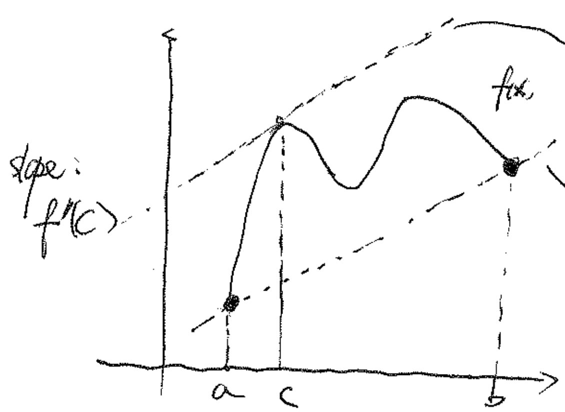
f is continuous and differentiable on $[-2,2]$.

MVT claims there exists $c \in [-2,2]$ such that

$f'(c) = \frac{f(2)-f(-2)}{2-(-2)} = 0$. Actually, we know in this example

$c=0$, since $f'(x) = (4-x^2)' = -2x \Rightarrow f'(0) = 0$.

Remark: $\frac{f(b)-f(a)}{b-a}$ is the slope of the straight line passing through $(a, f(a))$, $(b, f(b))$.
(secant)

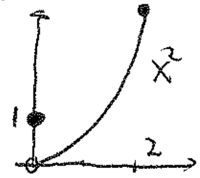


→ MVT claims there is a tangent line with the same slope as the secant line!

→ the secant line has slope $\frac{f(b)-f(a)}{b-a}$

eg2. Can the MVT be applied to the following functions? why?

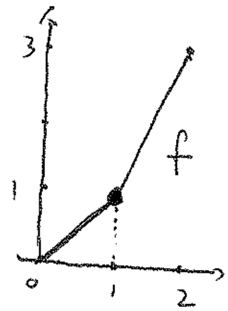
① $f(x)=\begin{cases}1 & x=0\\ x^2 & 0<x\le 2\end{cases}$ on $[0,2]$.



$f(x)$ has a break (jump) at $x=0$, ie., not continuous at $x=0$, ie., not continuous on $[0,2]$. (Closed interval)

No. MVT doesn't apply since f is not continuous on $[0,2]$

② $f(x)=\begin{cases}x & 0\le x<1\\ 2x-1 & 1\le x\le 2\end{cases}$ on $[0,2]$



f has a sharp turn at $x=1$, therefore, f is not differentiable at $x=1$, ie, not on $(0,2)$.

No. MVT doesn't apply since f is not differentiable on $(0,2)$.

eg3. (F16, MC) Consider the function $f(x)=x^2-4x$ on $[1,4]$. Apply MVT to f on $[1,4]$. Find the number c which satisfies the conclusion of the theorem.

Solution: $f(x)=x^2-4x$, on $[1,4]$. f is $\boxed{\text{continuous}}$ on $[1,4]$ and differentiable on $(1,4)$.

$a=1$, $b=4$. Conclusion: there is $c\in(1,4)$ such that $f'(c)=\dfrac{f(4)-f(1)}{4-1}$.

Goal: Compute $f'(x)$ and solve for c in the above equation.

$f'(x)=(x^2-4x)'=2x-4 \Rightarrow f'(c)=2c-4$

$f(4)=4^2-4\cdot 4=0$, $f(1)=1^2-4\cdot 1=-3 \Rightarrow \boxed{2c-4=\dfrac{0-(-3)}{4-1}}$

$2c-4=\dfrac{3}{3}=1 \Rightarrow 2c=5 \Rightarrow \boxed{c=\dfrac{5}{2}}$

eg.4 Find the number c that satisfies the conclusion of MVT for $f(x)=x^3+x$ on $[-1,0]$

Solution: $a=-1$, $b=0$, $f(-1)=(-1)^3-1=-2$, $f(0)=0$. $\dfrac{f(0)-f(-1)}{0-(-1)}=\dfrac{0-(-2)}{0-(-1)}=2$

$f'(x)=3x^2+1 \Rightarrow f'(c)=\boxed{3c^2+1=2}=\dfrac{f(0)-f(-1)}{0-(-1)}$

Solve for c: $3c^2=1$, $c^2=\frac{1}{3} \Rightarrow c=\pm\frac{1}{\sqrt{3}} \Rightarrow c=-\frac{1}{\sqrt{3}}\in(-1,0)$

(the positive $c=\frac{1}{\sqrt{3}}$ is discarded)

☆. §3.3. Derivatives and Graphs

Key points: ① Signs of $f'(x)$ and monotonicity (increasing/decreasing) of f.

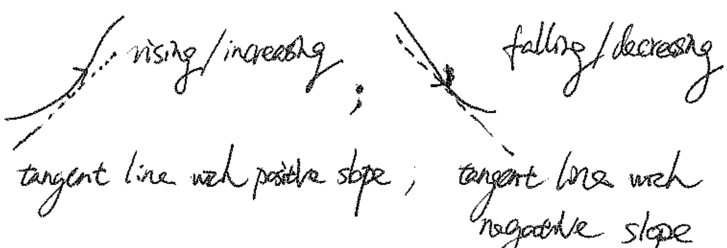
② Signs of $f''(x)$ and concavity (up/down) of f.

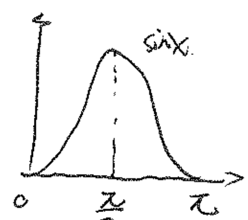
③ First/Second Derivative Test for local maximum/minimum.

④. Sketch the curve of $f(x)$ via $f'(x)$ and $f''(x)$. (signs).

Def: • $f(x)$ is increasing on $[a,b]$ if the GRAPH IS RISING, i.e., $f(x_1) < f(x_2)$ for all $x_1 < x_2$

• $f(x)$ is decreasing on $[a,b]$ if the GRAPH IS FALLING, i.e., $f(x_1) > f(x_2)$ for all $x_1 < x_2$





$\sin x$ is increasing on $[0, \frac{\pi}{2}]$

is decreasing on $[\frac{\pi}{2}, \pi]$

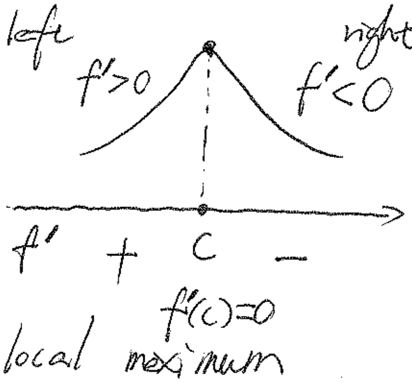
☆ Theorem: let $f(x)$ be continuous on $[a,b]$.

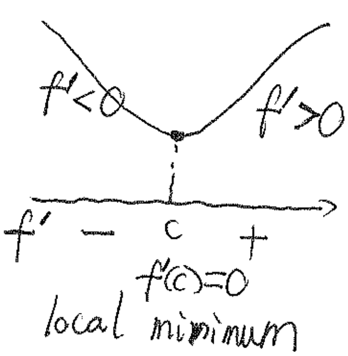
- If $f'(x) > 0$, then f is increasing.
- If $f'(x) < 0$, then f is decreasing.

Sign of f'	+	−	0
f	increasing	decreasing	critical point

• First Derivative Test (for local extremum): (c is a critical point of f), $f'(c)=0$.

If $f'(x)$ has different signs on the left and right hand sides of c, then $f(x)$ has a local extremum at $x=c$.





eg. 1 (F16). Suppose $f(x) = x^4 - 2x^2 - 3$. Find the intervals over which $f(x)$ is increasing and decreasing, and all values of x where $f(x)$ attains its local maximum or minimum.

Solution: $f'(x) = (x^4 - 2x^2 - 3)' = 4x^3 - 4x = 4x\cdot(x^2-1) = 4x\cdot(x+1)(x-1)$

Remark: We factorize $4x^3 - 4x = 4x(x+1)(x-1)$ since we want to determine f''s signs.

signs of f': $-$ (-1) $+$ (0) $-$ (1) $+$

$f'(-2)<0$, $f'(-0.5)>0$, $f'(0.5)<0$, $f'(2)>0$

f' has three zeros, $-1, 0, 1$, which break the ~~real~~ axis into four parts $(-\infty,-1)$, $(-1,0)$, $(0,1)$, $(1,+\infty)$. Plug in some simple numbers in each part to determine the sign.

$f' > 0 \Rightarrow f$ is increasing $\Rightarrow [-1,0] \cup [1,+\infty)$

$f' < 0 \Rightarrow f$ is decreasing $\Rightarrow (-\infty,-1] \cup [0,1]$

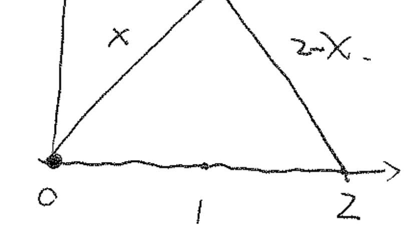
at $x=-1$, left $-$ right $+$: f has local minimum at $x=-1$.

at $x=0$, left $+$, right $-$: f has local maximum at $x=0$

at $x=1$, left $-$, right $+$: f has local minimum at $x=1$.

Remark: If the graph of $f(x)$ can be determined directly, then use graph to find the intervals / local extremums of f.

eg. 2. $f(x)$ on $[0,2]$ is defined as $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$



$f(x)$ is increasing on $[0,1]$

decreasing on $[1,2]$

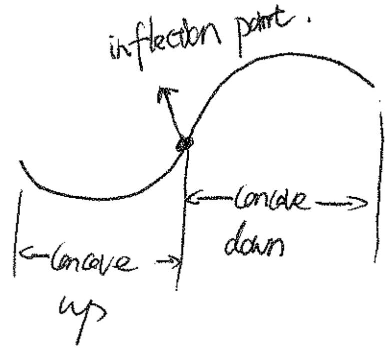
attains local (absolute) maximum at $x=1$

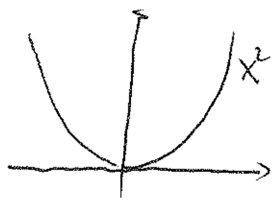
attains local minimum at $x=0$ and $x=2$.

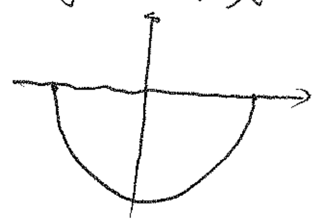
- Def (Concavity):
  - f is concave up if the graph is part of a smiling curve: 
  - f is concave down if the graph is part of a frowning curve: 

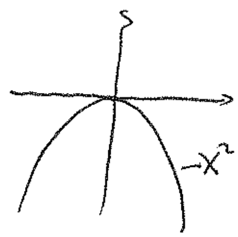
Concave up: 

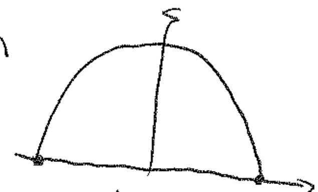
Concave down: 



- $y=x^2$ is concave up 

- Lower semi circle: $y=-\sqrt{1-x^2}$ is concave up 

- $y=-x^2$ is concave down 

- Upper semi circle $y=\sqrt{1-x^2}$ is concave down 

- Def: $f(x)$ has an inflection point at $x=c$ if $f'(x)$ has a local extremum at c.

  OR: c is an inflection point if f is concave up on one side of c and concave down on the other side

☆ Theorem:
- $f''(x)>0$ over (a,b), then $f(x)$ is concave up on (a,b)
- $f''(x)<0$ over (a,b), then $f(x)$ is concave down on (a,b)
- $f''(c)=0$ and $f''(x)$ has different signs on the two sides of c, then c is an inflection point.

signs of f''	$+$	$-$	0 (and changes signs)
f	concave up	concave down.	inflection point.

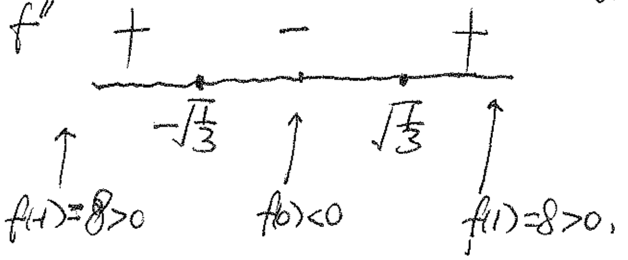
e.g.3. For $f(x)=x^4-2x^2-3$ in eg.1. Find where is f concave up/down and its inflection pts.

(☆ and then sketch the curve of $y=f(x)$.)

Solution: $f'(x)=4x^3-4x \Rightarrow \boxed{f''(x)=12x^2-4} = 4(3x^2-1)$

$= 4(\sqrt{3}x+1)(\sqrt{3}x-1)$

or $f''(x)=12(x^2-\frac{1}{3})=12\cdot(x+\sqrt{\frac{1}{3}})\cdot(x-\sqrt{\frac{1}{3}})$



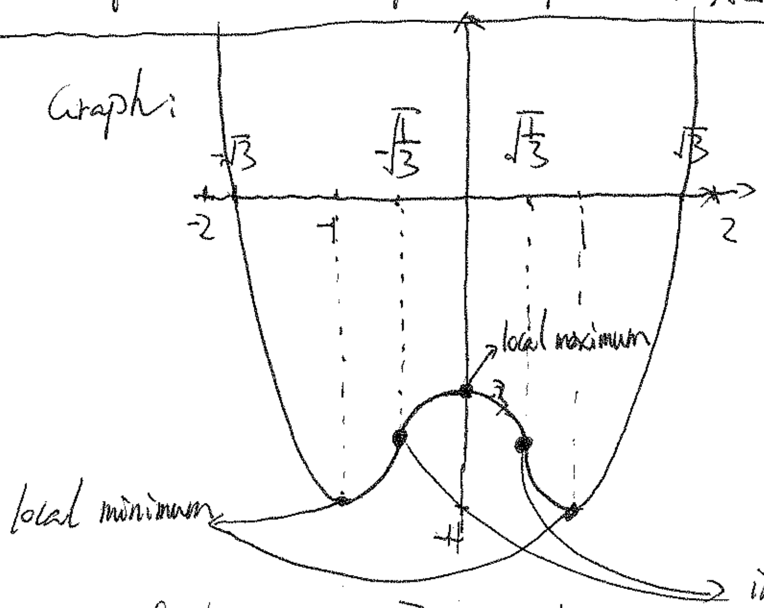
$f''(\pm\sqrt{\frac{1}{3}})=0$

Concave up: $f''>0$: $(-\infty,-\sqrt{\frac{1}{3}})\cup(\sqrt{\frac{1}{3}},+\infty)$

Concave down: $f''<0$: $(-\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}})$

f has two inflection points: $x=-\sqrt{\frac{1}{3}}$ and $x=\sqrt{\frac{1}{3}}$.

Graph:



y intercept: $f(0)=-3$.

☆ x intercepts: $f(x)=x^4-2x^2-3=0$

$\Leftrightarrow (x^2-3)\cdot(x^2+1)=0$

$\Leftrightarrow (x+\sqrt{3})(x-\sqrt{3})(x^2+1)=0$

x intercepts: $x=\pm\sqrt{3}$.

$f(\pm 1)=1-2-3=-4$.

$f(\pm\sqrt{\frac{1}{3}})=(\sqrt{\frac{1}{3}})^4-2\cdot(\sqrt{\frac{1}{3}})^2-3=-3\frac{5}{9}$

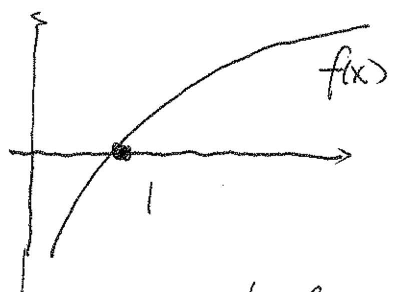
Second Derivative Test: Suppose $f'(c)=0$.

- $f''(c)<0$, then f attains a local maximum at $x=c$
- $f''(c)>0$, then f attains a local minimum at $x=c$
- $f''(c)=0$, then the second derivative test is inconclusive.

Remark: In most cases, the first derivative test will be enough for local extremums.

More examples from actual exams.

e.g. 4. (S16) The graph of f is given below. What can you say about the signs of $f(1)$, $f'(1)$, $f''(1)$? (Compare the largeness of them).



$\boxed{f(1)=0}$, (x intercept is 1)

$f(x)$ is increasing (near 1) $\Rightarrow \boxed{f'(1)>0}$

$f(x)$ is concave down $\Rightarrow \boxed{f''(1)<0}$

Therefore, $\boxed{f''(1)<f(1)<f'(1)}$.

e.g. 5 (S16). Find the inflection point (coordinates) for the function $f(x)=\sin x-\cos x$ in $[0,\pi]$

Solution: Inflection point $\Rightarrow f''=0$.

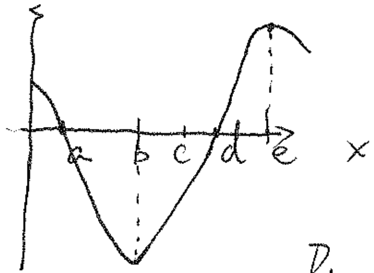
$$f'(x)=(\sin x-\cos x)'=(\sin x)'-(\cos x)'=\cos x-(-\sin x)=\cos x+\sin x.$$

$$\boxed{f''(x)=(\cos x)'+(\sin x)'=-\sin x+\cos x=0}$$

$$-\sin x+\cos x=0 \Rightarrow x=\frac{\pi}{4},\quad f(\tfrac{\pi}{4})=\sin\tfrac{\pi}{4}-\cos\tfrac{\pi}{4}=\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}=0$$

The inflection point at $x=\frac{\pi}{4}$ is $\boxed{(\frac{\pi}{4},0)}$.

eg.6. The graph of $f'(x)$ is given. At what value of x does
(F16) $f(x)$ has a local maximum?



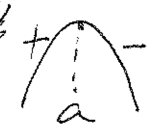
Caution: This is the graph $\boxed{\text{of } f'. \text{ NOT of } f}$

$x=e$ is not a local maximum.

Recall: local maximum $\Rightarrow f'=0$ and f' changes signs.

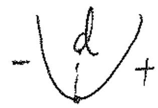
From the graph, $f'(a)=f'(b)=0$.

✓ $\boxed{x=a:}$ $f': + \to -$.



$\boxed{\text{local maximum}}$

$x=d$: $f': - \to +$



local minimum

Hints for webwork

ww10: $y=A\cdot x^{\frac{1}{4}}+B\cdot x^{-\frac{1}{4}}$ has an inflection point at $(1,6)$. Find A,B.

Plug in $x=1, y=6$. $6=A\cdot 1^{\frac{1}{4}}+B\cdot 1^{-\frac{1}{4}} \Rightarrow 6=A+B$.

Find y''. $y'=A\cdot\frac{1}{4}\cdot x^{-\frac{3}{4}}+B\cdot(-\frac{1}{4})\cdot x^{-\frac{5}{4}}$

$$y''=A\cdot\frac{1}{4}\cdot(-\frac{3}{4})\cdot x^{-\frac{7}{4}}+B\cdot(-\frac{1}{4})\cdot(-\frac{5}{4})x^{-\frac{9}{4}}$$
$$=-A\cdot\frac{3}{16}\cdot x^{-\frac{7}{4}}+B\cdot\frac{5}{16}\cdot x^{-\frac{9}{4}}$$

inflection point $\Rightarrow x=1, y''=0 \Rightarrow 0=-A\cdot\frac{3}{16}+B\cdot\frac{5}{16} \Rightarrow 0=-3A+5B$

$\begin{cases} 6=A+B \\ 0=-3A+5B \end{cases}$ solve for $A,B \Rightarrow A=\frac{5B}{3}$, $6=A+B=\frac{5B}{3}+B=\frac{8}{3}B$

$\Rightarrow \boxed{B=\frac{18}{8},\quad A=\frac{5}{3}\cdot\frac{18}{8}}$

ww7:

graph of $y=\frac{1}{x-5}$ $\Rightarrow$ graph of $y=\left|\frac{1}{x-5}\right|=\frac{1}{|x-5|}$ $\Rightarrow$ increasing on $(-\infty,5)$
decreasing on $(5,+\infty)$